

Energy transfer modelling of active thermoacoustic engines via Lagrangian thermoacoustic dynamics



Boe-Shong Hong*, Chia-Yu Chou

Department of Mechanical Engineering, National Chung Cheng University, Chia-Yi 621, Taiwan

ARTICLE INFO

Article history:

Received 1 July 2013

Accepted 4 April 2014

Available online 4 May 2014

Keywords:

Thermoacoustic engines
nD transfer function models
Thermoacoustic dynamics
Lagrange dynamics
Rayleigh criterion
Robin boundary conditions

ABSTRACT

This paper develops energy-transfer modelling of active thermoacoustic engines resonantly controlled on boundary for amplification of power rating toward satisfaction of renewable industry. Therein the wave equation of thermoacoustic dynamics in resonators with non-uniform media and boundary actuations is derived and then turned into a least-action principle. With this least-action principle, we obtain the governing equation of longitudinal resonators with spatially variant cross-section areas to investigate how to shape the resonator for boosting piston stroke and power-transmission efficiency. It is followed by spatiotemporal transfer-function modelling that functionally represents the dynamics and interprets the boundary actuations into internal inputs. This helps formulate the overall dynamics into feedback-interconnection between the thermoacoustic dynamics in the resonator and the mechatronic dynamics of the alternative current generator, so that synthesis of feedback systems can be applied to design the entire engine. Transfer-function modelling following least-action principle leads to the conservation law of thermoacoustic storage, which figures out engine cycles, the most fundamental principle in designing active thermoacoustic engines. Based on such feedback realization, digital signal processing is programmed to numerically assess power ratings of active designs.

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1. Introduction

Of facilities generating electricity from sunlight, the thermoacoustic engine is of much less ratio of manufacturing cost to power rating than photovoltaic panels [1,2], solar-thermal devices, and other heat engines. Its extremely simple mechanism enables us to build solar power plants in poor lands and home rooftops at affordable prices. Besides, thermoacoustic engines under resonant control can have power efficiency even better than Carnot cycle efficiency that merely exists in an ideal heat engine. Inasmuch as passive thermoacoustic engines, for examples in [3–10], poorly generate electricity, this work aims to investigate active thermoacoustic engines with power ratings toward the requirement of renewable energy industry.

The active thermoacoustic engine under design is assembled by alternative current generator, Rijke tube, and radiation-to-heatflux actuator, as shown in Fig. 1. The radiation-to-heatflux actuator is comprised of crystal liquid photo valve and ultra-conducting porous media adjacent to each other. It consumes very little power to actively switch the heat-flux source at resonant frequencies

for amplification of power rating. At the head end of the Rijke tube, the heat-flux resulting from sunlight gives rise to entropy rate that changes the fluid density there, which in turns initiates acoustic velocity. That is, the heat-flux excitation expanding and contracting the fluid acts as a source of acoustic wave that rides on Rijke-tube media and arrives at the load end to push and pull the piston of alternative current generator.

A frequency generator embedded therein is programmed to online keep the switching frequency of the buck chopper that drives the photo valve on one of resonant frequencies, which prompts standing waves up to resonance and thus increases mechanical power enormously. Besides, the Rijke tube is longitudinally shaped to make power transmitted from the heatflux actuator to the current generator be with high pressure and low velocity. Such high-pressure AC power transmission reduces the viscosity effect inside the Rijke tube and increases the piston stroke, analogously to adjust the gear ratio in mechanical transmission. The stackers inside the Rijke tube are for internal heating of working gas to maintain higher-frequency resonance for larger power rating. For the piston at the tail end to move back and forth, some clearance between the Rijke tube and the piston surface produces acoustic leakage and friction on the right boundary. We model this practical situation by Robin boundary condition.

* Corresponding author. Tel.: +886 5 2720411x33321.

E-mail address: imehbs@ccu.edu.tw (B.-S. Hong).

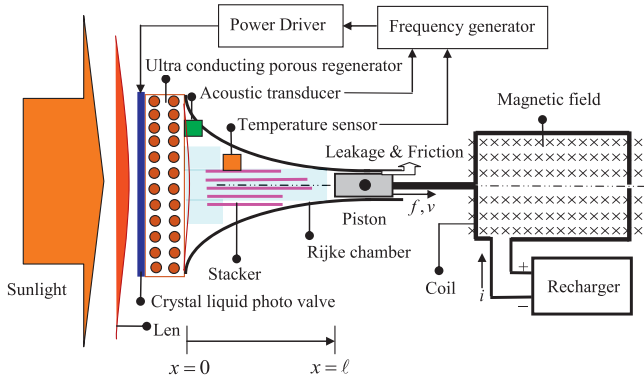


Fig. 1. Schematic design of resonant thermoacoustic engines.

The thermoacoustic dynamics inside Rijke tubes has been intensively studied in recent years, for examples, based on impedance analysis [6–10], computational fluid dynamics [11–14] or spatio-temporal transfer-function [15]. In fact, these studies mostly fall heirs to the thermoacoustic dynamics in the field of combustion instabilities, for examples in [16–20]. No matter in thermoacoustic engines or in combustion instabilities, the kernelled dynamics is concerned with the process how heat perturbation excites acoustic motions, known as *Rayleigh Criterion* [15,21–24]. In this paper, the Rayleigh criterion is further formulated into a *principle of least action*, named here by *Lagrangian thermoacoustic dynamics*. It is then employed to derive the wave equation governing energy transfer through Rijke tubes with spatially variant cross-section areas.

With Lagrangian formulation, cylinder's pressure-velocity amplitudes are thus capable of being longitudinally adjusted mainly for energy efficiency in power transmission. As the cross-section is nearly exponentially narrowed along the longitudinal direction, power is transmitted from the head end with larger pressure and tiny velocity to the load end with smaller pressure and required velocity. Low-velocity AC-power transmission reduces the viscosity effect inside the Rijke tube and thus increases energy efficiency.

The wave equation with non-uniform media in conjunction with inhomogeneous boundary conditions is then transformed into a single spatiotemporal transfer function, which is an expanded application of the functional tool known as nD transfer function models [25–29]. Therein the inhomogeneous boundary conditions arise from the von Neumann source of heat-flux at the head end and the Robin source of piston-speed at the tail end, both of which are transformed into Dirac internal sources by way of Laplace–Galerkin transform. This renders the capability of representing the overall dynamics into feedback-interconnection between the dynamics of the Rijke tube dynamics and the alternative current generator, and thus of transforming the engine design into synthesis of feedback systems. Moreover, digital signal processing can be programmed based on this feedback realization to offline or online assess the power rating.

A new conservation law is inferred from transfer-function transformation of Lagrangian thermoacoustic dynamics, named by *conservation law of thermoacoustic storage*. It figures out engine cycles and thus is the most fundamental principle in designing active thermoacoustic engines. For example, flow leakage through the gap between piston surface and chamber interior happens in real designs, which results in a Robin source to the load end. Fortunately, we can estimate the level of flow leakage with the conservation law of thermoacoustic storage through identification of Robin coefficients therefrom. As we know, Robin coefficients can only be determined through system identification [30,31], and this paper provides a method to identify the Robin coefficients of flow leakage.

2. Lagrangian thermoacoustic dynamics

Let Ω be a Rijke tube containing non-uniform gas medium with time-invariant, spatial variant entropy $\bar{s}$, pressure $\bar{p}$, temperature $\bar{T}$, and density $\bar{\rho}$. Imposed on the medium is the thermoacoustic motion with entropy fluctuation s , acoustic pressure p , and acoustic velocity u . The following assumptions are made:

- (1) The medium is an ideal gas.
- (2) The heat transfer and viscosity in the field of fluctuation is neglected.
- (3) The medium is stationary, i.e., $\bar{u} = 0$.

The dynamics of thermoacoustic motion was usually derived from the first-order perturbation of fluid conservation equations [32]. Application of thermoacoustic state transformations in [33] to this set of perturbation equations yields a new phrase as follows:

$$\frac{\partial s}{\partial t} = -\nabla \bar{s} \cdot u + \frac{Q}{\bar{\rho} \bar{T}}, \quad (1)$$

$$\frac{\partial u}{\partial t} = -\frac{1}{\bar{\rho}} \nabla p, \quad (2)$$

$$\frac{\partial p}{\partial t} = -\kappa \nabla \cdot (\bar{\rho} u) + (\gamma - 1)(Q - \bar{\rho} \bar{T} \nabla \bar{s} \cdot u), \quad \kappa = \gamma R \bar{T}, \quad (3)$$

where Q stands for the source of heat-perturbation rate. Therein the parameters R , γ and κ are the gas constant, the specific heat ratio and the compressibility of the gas medium, respectively.

Eq. (1) implies that spatial variation of mean entropy $\bar{s}$ on the border of any differential volume moving with bulk velocity u ($\bar{u} = 0$) produces entropy flux $-\nabla \bar{s} \cdot u$ through the border. The total entropy-rate of the differential volume is the entropy flux $-\nabla \bar{s} \cdot u$ plus the entropy change $Q/\bar{\rho} \bar{T}$ provided by the heat-perturbation rate Q . Eq. (2) is directly from the Newton's second law. Eq. (3) is based on the state transformation:

$$\frac{p}{\gamma \bar{p}} = \frac{\rho}{\bar{\rho}} + \frac{s}{\bar{c}_p}, \quad (4)$$

Combined by the law of continuity:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\bar{\rho} u) \quad (5)$$

Eq. (4) results from the linearized Gibbs equation in conjunction with the linearized state equation for an ideal gas. It implies that “pressure change can be produced by fluid compression or entropy fluctuation” [33]. Substituting Eq. (1) and the temporal differential of Eq. (5) into Eq. (4) yields Eq. (3).

Then, substituting Eq. (2) into the temporal differential of Eq. (3) yields

$$\frac{\partial^2 p}{\partial t^2} - \kappa \nabla^2 p = -(\gamma - 1) \bar{\rho} \bar{T} \nabla \bar{s} \cdot \frac{\partial u}{\partial t} + (\gamma - 1) \frac{\partial Q}{\partial t} \quad (6)$$

Defining the *thermoacoustic momentum* ψ by the temporal integration of acoustic pressure p , that is,

$$p = \frac{\partial \psi}{\partial t}; \quad \bar{\rho} u = -\nabla \psi, \quad (7)$$

We rephrase Eq. (6) by

$$\frac{\partial^2 \psi}{\partial t^2} - (\kappa \nabla + (\gamma - 1) \bar{T} \nabla \bar{s}) \cdot \nabla \psi = (\gamma - 1) Q, \quad \text{in } \Omega. \quad (8)$$

Firstly, Eq. (8) can be transformed to a wave equation free of damping:

$$\frac{\partial^2 \psi}{\partial t^2} - \frac{\kappa}{e^{(\tilde{s}-\tilde{s}_0)/C_p}} \nabla \cdot (e^{(\tilde{s}-\tilde{s}_0)/C_p} \nabla \psi) = (\gamma - 1)Q, \quad (9)$$

where the $\tilde{s}_0$ is the mean-entropy at an arbitrarily chosen location x_0 inside or outside the chamber Ω . Therein the parameter C_p is the constant-pressure heat capacitance. The derivation from Eq. (8) of Eq. (9) is as follows. Eq. (8) is identical to

$$\frac{\partial^2 \psi}{\partial t^2} - \kappa \left(\nabla^2 \psi + \frac{\nabla \tilde{s}}{C_p} \cdot \nabla \psi \right) = (\gamma - 1)Q, \quad C_p = \frac{\gamma R}{\gamma - 1}.$$

In juxtaposition of this equation with the canonical form of wave equation, such as Eq. (9), it is found that the existence of an integration factor σ such that

$$\sigma((\nabla \tilde{s}/C_p) \cdot \nabla \psi + \nabla^2 \psi) = \nabla \cdot (\sigma \nabla \psi)$$

is necessary to transform Eq. (8) to Eq. (9). The above equation implies $\nabla \sigma / \sigma = \nabla \tilde{s} / C_p$, that is, $\sigma = \exp\left(\frac{1}{C_p} \int_C \nabla \tilde{s} \cdot d\mathbf{r}\right)$, where C is an arbitrary path for line integral, wherein the line integral is independent of path since the mean-entropy $\tilde{s}$ belongs to thermodynamic state. Therefore such a factor σ exists, which equals $\sigma = e^{(\tilde{s}-\tilde{s}_0)/C_p}$. With this found factor, it can be verified that Eq. (8) is identical to Eq. (9).

Secondly, the state equation combined by the first law of thermodynamics [33] equates

$$e^{(\tilde{s}-\tilde{s}_0)/C_p} = \frac{\tilde{T}}{\tilde{T}_0} \left(\frac{\tilde{p}}{\tilde{p}_0} \right)^{\frac{1-\gamma}{\gamma}} \quad (10)$$

where $\tilde{T}_0$ and $\tilde{p}_0$ are the mean temperature and pressure, respectively, at the same reference location x_0 . Substitution of Eq. (9) for Eq. (10) yields

$$\frac{\partial^2 \psi}{\partial t^2} - \gamma R \tilde{T}_0 \left(\frac{\tilde{p}}{\tilde{p}_0} \right)^{\frac{1-\gamma}{\gamma}} \nabla \cdot \left(\frac{\tilde{T}}{\tilde{T}_0} \left(\frac{\tilde{p}}{\tilde{p}_0} \right)^{\frac{1-\gamma}{\gamma}} \nabla \psi \right) = (\gamma - 1)Q. \quad (11)$$

It can be verified that $\tilde{T}_0$ and $\tilde{p}_0$ in Eq. (11) can be removed without any change, which is in line with the fact that the dynamics has to be independent of the choice of reference location x_0 . Defining the dimensionless Rayleigh factor by $R_a \equiv (\tilde{p}/\tilde{p}_0)^{(1-\gamma)/\gamma}$, we express the thermoacoustic dynamics by

$$\frac{\partial^2 \psi}{\partial t^2} - \frac{1}{R_a} \nabla \cdot (\kappa R_a \nabla \psi) = (\gamma - 1)Q, \quad (12)$$

where $\kappa = \gamma R \tilde{T}$ denotes the compressibility. Without losing generality, the reference pressure $\tilde{p}_0$ can be chosen as the environmental pressure.

To realize the dynamic behaviour described in Eq. (12), Rayleigh [34] claimed the analogy of acoustic pressure p to mechanical velocity and heat-rate source Q to excitation force, which has been largely investigated in combustion instabilities earlier or thermoacoustic engines later as the Rayleigh criterion. To quantify this analogy based on Eq. (12), we define two internal scalars: Rayleigh's Kinetics K_R , Rayleigh's Potential P_R and one line integral Rayleigh's Work W_R respectively by

$$2K_R \equiv \int_{\Omega} R_a \left(\frac{\partial \psi}{\partial t} \right)^2 dV, \quad 2P_R \equiv \int_{\Omega} \kappa R_a |\nabla \psi|^2 dV, \quad dW_R \equiv \int_{\Omega} d\psi \cdot R_a (\gamma - 1)Q dV. \quad (13)$$

With calculus of variation, Eq. (12) leads to a least-action principle:

$$0 = \int_{t_1}^{t_2} \delta K_R - \delta P_R + \delta W_R dt, \quad (14)$$

with respect to the variation of Rayleigh's displacement $\delta \psi$, where δ represents the operator of functional variation. It can be verified

that calculus of variation upon Eq. (14) also leads back to the wave equation of Eq. (12), named here by Lagrangian thermoacoustic dynamics.

3. Spatiotemporal transfer function modelling

Fig. 1 shows the thermoacoustic engine schematics under design that comprises that comprises the Rijke chamber, the radiation-to-heatflux transducer, the photo valve, and the alternative current generator. Here we consider the thermoacoustic dynamics to be longitudinal in the Rijke tube with spatially variant temperature, pressure and cross-section area. Moreover, the flow leakage and boundary friction are also concerned.

The heat-flux excitation q (W/m²) on the head end ($x = 0^-$) activating thermoacoustic process inside the Rijke chamber is equivalent to a pointed heat generation $q\delta(x)$ between $x = 0^-$ and $x = 0^+$ with heat isolation at $x = 0^-$, based on the localization of energy conservation. Here δ is the Dirac delta distribution. With Eq. (13), the Rijke chamber stores Rayleigh's Kinetics K_R and Rayleigh's Potential P_R being, respectively,

$$2K_R \equiv \int_0^\ell R_a \dot{\psi}^2 A dx \quad \text{and} \quad 2P_R \equiv \int_0^\ell \kappa R_a \psi'^2 A dx, \quad (15)$$

where A stands for the cross-section area of the Rijke chamber that is spatially variant. The Lagrange formulation of the Least-action principle of Eq. (14) upon Eq. (15) yields the following longitudinal dynamics:

$$\frac{\partial^2 \psi}{\partial t^2} - \frac{1}{R_a A} \frac{\partial}{\partial x} \left(\kappa R_a A \frac{\partial \psi}{\partial x} \right) = (\gamma - 1)q\delta(x) \quad \text{in } [0, \ell], \quad (16)$$

in addition to piston vibration at $x = \ell$. Since the reference pressure $\tilde{p}_0$ has been chosen as the environmental pressure, the Rayleigh factors on both boundaries are of unity, i.e., $R_a(0) = R_a(\ell) = 1$. Reduction of non-uniform medium to be spatially invariant and replacement of heat-flux excitation by acoustic driver, Eq. (16) appears to be Webster's horn equation which is well-known in the acoustics of loudspeakers and flared wind and brass instruments.

Eq. (16) facilitates the investigation of the thermoacoustic process happened on the head end between $x = 0^-$ and $x = 0^+$. The hyperbolic Eq. (16) has signified that the acoustic pressure p in Rijke chamber is spatially continuous at $x = 0$, i.e., there is no pressure jump from $x = 0^-$ to $x = 0^+$. Thus, the integration of Eq. (16) between $x = 0^-$ and $x = 0^+$ yields the left boundary condition to be

$$C_p \tilde{T} \psi' = -q, \quad \text{at } x = 0^+, \quad (17)$$

where $C_p = R\gamma/(\gamma - 1)$. Based on the definition in Eq. (7), Eq. (17) can be rephrased by

$$u = \frac{\gamma - 1}{\gamma} \frac{q}{\tilde{p}_0} \quad \text{at } x = 0^+ \quad (18)$$

Eq. (18) reveals that the heat-flux q initiates an acoustic velocity on the left boundary, but keeps the acoustic pressure continuous on the boundary.

The dynamic process on the head end as indicated by Eq. (18) can be realized by tracing back to the Chu's formula of Eq. (4) and the continuity law of Eq. (5) as follows. In between $x = 0^-$ and $x = 0^+$ the heat-flux excitation q gives rise to entropy rate $\dot{s}$ that changes the fluid density, as indicated by Eq. (4), which in turns initiates acoustic velocity, as indicated in Eq. (5). That is, the heat-flux excitation expanding and contracting in fast-time scale the fluid on the boundary $x = 0$ acts as a source of acoustic wave. The acoustic wave rides on Rijke chamber media and arrives at the load end at $x = \ell$ to push and pull the piston, generating mechanical work.

As for the boundary condition at the load end, it is nominally taken to be

$$\psi' = -\tilde{\rho}v, \text{ at } x = \ell^-,$$

with the assumption that the acoustic pressure is continuous at $x = \ell^-$ and there is no heat-flux excitation at $x = \ell^+$, where v stands for the piston speed. If we take flow leakage and boundary friction into consideration, the right boundary condition belongs empirically to the Robin type:

$$\alpha\psi + \beta\psi' = -\tilde{\rho}v, \text{ at } x = \ell^-, \quad (19)$$

where $\alpha > 0$, $\beta > 0$. Here the leakage factor β concerns the ratio of closed area to open area on the load-end boundary, which is to be identified in real design. Moreover, the piston dynamics chosen for representation is governed by

$$m\dot{v} + bv = \dot{\psi}_\ell, \quad (20)$$

wherein m stands for the inertia of piston and b comprises the boundary and load damping. For succinct notation, $\psi(x, t)$ may appear as $\psi_x(t)$ somewhere in this paper and similarly to other variables.

The collection of Eq. (16) to Eq. (20) describes the thermoacoustic engine dynamics as

$$\frac{\partial^2 \psi}{\partial t^2} - \frac{1}{R_a A} \frac{\partial}{\partial x} \left(\kappa R_a A \frac{\partial \psi}{\partial x} \right) = 0, \text{ in } (0, \ell); \quad (21)$$

$$C_p \tilde{T} \psi' = -q, \text{ at } x = 0, \quad (22)$$

$$\alpha\psi + \beta\psi' = -\tilde{\rho}v, \text{ at } x = \ell (\alpha > 0, \beta > 0); \text{ and} \quad (23)$$

$$m\dot{v} + bv = \dot{\psi}, \text{ at } x = \ell, \quad (24)$$

wherein the Rijke-tube dynamics of Eqs. (21)–(23) is feedback-coupled with the represented load dynamics in Eq. (24).

The Rijke-chamber dynamics in Eqs. (21)–(23) involves the elastic stiffness operator $\mathcal{K}$:

$$\mathcal{K}\phi = -\frac{1}{R_a A} \frac{\partial}{\partial x} \left(\kappa R_a A \frac{\partial \phi}{\partial x} \right),$$

the domain of which is

$$D(\mathcal{K}) \equiv \{\phi : \phi' = 0 \text{ at } x = 0; \alpha\phi + \beta\phi' = 0 \text{ at } x = \ell\},$$

where $\alpha > 0$, and $\beta > 0$. The stiffness operator $\mathcal{K}$ is self-adjoint and positive-definite under the inner-product:

$$\langle \varphi, \phi \rangle \equiv \int_0^\ell R_a(x) A(x) \phi^*(x) \varphi(x) dx. \quad (25)$$

Denote by Λ the eigenvalues set of the stiffness operator $\mathcal{K}$, corresponding to the eigenfunctions set $\Phi \equiv \{\phi_\lambda\}_{\lambda \in \Lambda}$, i.e.,

$$\mathcal{K}\phi_\lambda = \lambda\phi_\lambda \text{ in } [0, \ell], \quad \phi_\lambda \in D(\mathcal{K}) \text{ for } \forall \lambda \in \Lambda.$$

Accompanied by $\mathcal{K}$'s self-adjointness is that its eigenfunctions set Φ is real and orthonormal, and followed by $\mathcal{K}$'s positive-definiteness is that all of its eigenvalues are positive, $\lambda > 0$ for $\forall \lambda \in \Lambda$. Moreover, the countable eigenfunctions set Φ is a complete basis of $L_2([0, \ell])$, which follows that the inverse of $\mathcal{K}$ is a bounded operator and closed in the Banach space $L_2[0, \ell]$.

With respect to the eigenfunctions set Φ and the inner-product of Eq. (25), a 2D transform $\mathcal{H}$, named by *Laplace–Galerkin transform*, is then defined by the composite of Laplace transform and Galerkin transform $\mathcal{H} = \mathcal{L}\mathcal{G} = \mathcal{G}\mathcal{L}$, i.e.,

$$F(\lambda, s) \equiv \mathcal{H}[f(x, t)] = \int_{t=0}^\infty \int_{x=0}^{\ell} e^{-st} R_a(x) A(x) \phi_\lambda(x) f(x, t) dx dt, \quad (26)$$

to describe f 's modal decomposition in the s -domain. Accordingly, the inverse Laplace–Galerkin transform is the composite of the inverses of those two 1D transforms, i.e., $\mathcal{H}^{-1} = \mathcal{G}^{-1}\mathcal{L}^{-1} = \mathcal{L}^{-1}\mathcal{G}^{-1}$. Explicitly,

$$f(x, t) \equiv \mathcal{H}^{-1}[F(\lambda, s)] = \frac{1}{2\pi j} \sum_{\lambda \in \Lambda} \phi_\lambda(x) \int_\Gamma e^{ts} F(\lambda, s) ds. \quad (27)$$

With the above theoretical background, in the next paragraph we will perform the Laplace–Galerkin transform on Eqs. (21)–(24) to derive the 2D transfer-function modelling of the engine dynamics.

At the load end $x = \ell$, substitution of $\alpha\phi_\lambda + \beta\phi'_\lambda = 0$ for Eq. (23) $\times (-\phi_\lambda)$ yields

$$(\psi\phi'_\lambda - \phi_\lambda\psi') = (\phi_\lambda/\beta)\tilde{\rho}v \text{ at } x = \ell, \quad (28)$$

and the integration by parts yields

$$\int_0^\ell -\frac{\partial}{\partial x} \left(\kappa R_a A \frac{\partial \psi}{\partial x} \right) \cdot \phi_\lambda dx = \int_0^\ell -\frac{\partial}{\partial x} \left(\kappa R_a A \frac{\partial \phi_\lambda}{\partial x} \right) \cdot \psi dx + \kappa A \psi' \phi_\lambda|_{x=0} + \kappa A (\psi \phi'_\lambda - \phi_\lambda \psi')|_{x=\ell}.$$

Based on Eq. (22) and (28), the above equation becomes

$$\int_0^\ell -\frac{\partial}{\partial x} \left(\kappa R_a A \frac{\partial \psi}{\partial x} \right) \cdot \phi_\lambda dx = \lambda \langle \psi, \phi_\lambda \rangle - (\gamma - 1) q A_0 \cdot \phi_\lambda(0) + \beta^{-1} \gamma \tilde{p}_0 A_\ell v \cdot \phi_\lambda(\ell), \quad (29)$$

where $\tilde{p}_0$ denotes the environmental pressure, and therefore $R_a(0) = R_a(\ell) = 1$.

With Eq. (29), performing Laplace–Galerkin transform $\mathcal{H}$ on Eq. (21) yields

$$(s^2 + \lambda) \Psi(\lambda, s) = (\gamma - 1) A_0 \phi_\lambda(0) \cdot \hat{q}(s) - \tilde{p}_0 A_\ell \beta^{-1} \gamma \phi_\lambda(\ell) \cdot \hat{v}(s), \quad (30)$$

where $\Psi(\lambda, s) \equiv \mathcal{H}[\psi(x, t)]$, $\hat{q}(s) \equiv \mathcal{L}[q(t)]$, and $\hat{v}(s) \equiv \mathcal{L}[v(t)]$. The 2D transfer function of the Rijke-chamber dynamics of Eqs. (21)–(23) is thus obtained to be

$$\Psi(\lambda, s) = (\gamma - 1) A_0 \frac{\phi_\lambda(0) \hat{q}(s)}{s^2 + \lambda} - \tilde{p}_0 A_\ell \beta^{-1} \gamma \frac{\phi_\lambda(\ell) \hat{v}(s)}{s^2 + \lambda}, \quad (31)$$

feedback-coupled with the load dynamics:

$$\hat{v}(s) = \frac{s}{ms + b} \hat{\psi}_\ell(s), \quad (32)$$

which is obtained by performing Laplace transform on Eq. (24).

Eqs. 31, 32 constitute the spatiotemporal transfer-function modelling of the thermoacoustic engine. This represents the engine dynamics as a feedback-interconnection of the chamber dynamics in Eq. (31) and the load dynamics in Eq. (32), which transforms the overall design into synthesis of feedback systems. Moreover, the feedback-coupled transfer-functions can be realized in the state space for easily developing digital signal processing for online or offline computation of engine performance.

4. Conservation of thermoacoustic storage

Performing the inverse Laplace–Galerkin transform $\mathcal{H}^{-1}$ on the spatiotemporal transfer-function of Eq. (31) yields

$$\frac{\partial^2 \psi}{\partial t^2} - \frac{1}{R_a A} \frac{\partial}{\partial x} \left(\kappa R_a A \frac{\partial \psi}{\partial x} \right) = (\gamma - 1) q(t) A_0 \sum_{\lambda \in \Lambda} \phi_\lambda(0) \phi_\lambda(x) - \tilde{p}_0 \beta^{-1} \gamma v(t) A_\ell \sum_{\lambda \in \Lambda} \phi_\lambda(\ell) \phi_\lambda(x), \quad (33)$$

with homogeneous boundary conditions: $\psi'(0) = 0$ and $\alpha\psi(\ell) + \beta\psi'(\ell) = 0$. Eq. (33) equals

$$\frac{\partial^2 \psi}{\partial t^2} - \frac{1}{R_a A} \frac{\partial}{\partial x} (\kappa R_a A \frac{\partial \psi}{\partial x}) = (\gamma - 1) q(t) \delta(x) - \tilde{p}_0 \beta^{-1} \gamma v(t) \delta(x - \ell). \quad (34)$$

Eventually, the wave equation of Rijke tube in Eqs. (21)–(23) is mathematically identical to Eq. (34) with homogeneous boundary conditions. In this way, spatiotemporal transfer-function enables us to realize inhomogeneous boundary conditions as internal sources in conjunction with boundary homogeneity.

The operator form of Eq. (34) becomes

$$(\mathcal{M}\ddot{\psi} + \mathcal{K}\psi)(x, t) = (\gamma - 1) q(t) \delta(x) - \tilde{p}_0 \beta^{-1} \gamma v(t) \delta(x - \ell), \quad (35)$$

where the mass operator $\mathcal{M}$ is the identity operator, and the stiffness operator $\mathcal{K}$ that has been detailed in Section 3 is

$$\mathcal{K}\psi = -\frac{1}{R_a A} \frac{\partial}{\partial x} \left(\kappa R_a A \frac{\partial \psi}{\partial x} \right), \quad (36)$$

with the domain of operation

$$D(\mathcal{K}) \equiv \{\psi : \phi'(0) = 0, \quad \alpha\phi(\ell) + \beta\phi'(\ell) = 0\}.$$

Both operators are positive-definite and self-adjoint under the inner product $\langle \cdot, \cdot \rangle$ defined in Eq. (25).

Define a new quantity E , named by *thermoacoustic storage*, to be

$$2E = \langle \dot{\psi}, \mathcal{M}\dot{\psi} \rangle + \langle \psi, \mathcal{K}\psi \rangle. \quad (37)$$

The thermoacoustic storage E is positive-definite, since the operators $\mathcal{M}$ and $\mathcal{K}$ are positive-definite under the inner-product defined in Eq. (25). It becomes zero if and only if $\psi = 0$ and $\dot{\psi} = 0$ almost everywhere. The operators $\mathcal{M}$ and $\mathcal{K}$ are self-adjoint, so

$$\begin{aligned} \frac{dE}{dt} &= \frac{1}{2} (\langle \ddot{\psi}, \mathcal{M}\dot{\psi} \rangle + \langle \dot{\psi}, \mathcal{M}\ddot{\psi} \rangle + \langle \dot{\psi}, \mathcal{K}\psi \rangle + \langle \psi, \mathcal{K}\dot{\psi} \rangle) \\ &= \langle \dot{\psi}, \mathcal{M}\ddot{\psi} \rangle + \langle \dot{\psi}, \mathcal{K}\psi \rangle = \langle \dot{\psi}, \mathcal{M}\ddot{\psi} + \mathcal{K}\psi \rangle. \end{aligned} \quad (38)$$

Based on Eq. (35), the above equation becomes

$$\dot{E} = A_0(\gamma - 1)q p_0 - \tilde{p}_0 A_\ell \beta^{-1} \gamma v p_\ell, \quad (39)$$

where $p = \dot{\psi}$ denotes the acoustic pressure.

Further to define E_T by scaling the thermoacoustic storage E to be

$$E_T \equiv \frac{E}{\gamma \tilde{p}_0} \quad (40)$$

so that Eq. (38) becomes

$$\dot{E}_T = A_0 \frac{(\gamma - 1)q}{\gamma \tilde{p}_0} p_0 - \beta^{-1} A_\ell v p_\ell \quad (41)$$

Based on Eq. (18), the first term of the right side of Eq. (40) equals $A_0 u_0 p_0$, where u_0 is the acoustic velocity at the head end $x = 0$. Then Eq. (40) can be written by

$$\dot{E}_T = A_0 u_0 p_0 - \beta^{-1} A_\ell u_\ell p_\ell,$$

which reveals that E_T defines the mechanical energy stored in the thermoacoustic dynamics. The mechanical energy E_T extends the mechanical part of thermoacoustic energy for non-uniform pressure and temperature from that in uniform mean-flow conditions that has been defined in literature of thermoacoustic dynamics, for example, in [33].

Suppose the boundary heat-flux q entering into the chamber is of resonant frequency ω , in steady state the thermoacoustic storage E will be conserved over a period $T = 2\pi/\omega$, that is,

$$A_\ell \int_T v(t) p_\ell(t) dt = \beta A_0 \frac{(\gamma - 1)}{\gamma \tilde{p}_0} \int_T q(t) p_0(t) dt, \quad (42)$$

where the left side is the mechanical work generated over a period. There, the leakage factor $\beta < 1$ quantifies the loss of work in practice,

which is to be identified by Eq. (41). Furthermore, the integrals of Eq. (41) can be written as the integration over a cycle of thermoacoustic momentum ψ , that is,

$$\int_W v d\psi_\ell = \beta \frac{A_0}{A_\ell} \frac{(\gamma - 1)}{\gamma \tilde{p}_0} \int_S q d\psi_0, \quad (43)$$

where W and S denote the work cycle and source cycle, respectively.

Over a cycle, the points of (ψ_ℓ, v) depicts a closed region, the area of which indicates the mechanical energy generated or lost over a cycle. If the points of (ψ_ℓ, v) direct clockwise along the work cycle, it means that positive power is gained. Similarly, if the points of (ψ_0, u_0) direct clockwise along the source circle, the useful power excited by the boundary heat-flux is positive. As such, Eq. (42) defines the engine cycles for active thermoacoustic engines, as shown in Fig. 2, where the upper parts of the regions enveloped by (ψ_ℓ, v) and (ψ_0, u_0) indicate the expansion stroke, and the lower parts indicates contraction stroke. Unlike Stirling engines that have four strokes including heating and cooling strokes, this thermoacoustic engine has only two strokes so that they can be fulfilled by a single cylinder.

Engine cycles in Eq. (42) reveals the following design principles:

- (1) The heat-flux excitation can be on-line controlled to be of a frequency identical to one of the resonant frequencies to arrive at the maximum power generation.
- (2) The power rating of engine is proportional to the square of $(\gamma - 1)/\gamma$, concerned with the working gas.

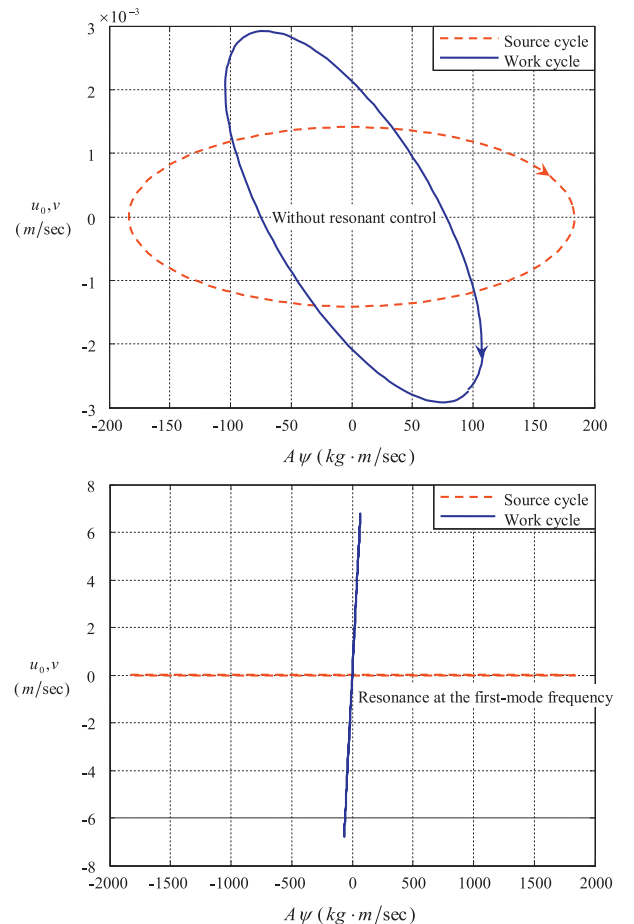


Fig. 2. Comparison of engine cycles at resonance with those without resonant control.

- (3) The ratio of the heat-end area A_0 to the load-end area A_ℓ plays like a gear ratio in power transmission adaptable to the mechatronic load.
- (4) The engine is getting more powerful along with higher working frequency of resonance that is usually constrained by control rig and working temperature.

5. Performance review

Let the thermoacoustic engine of Fig. 1 be with solar heat-flux excitation $q = 500 \text{ W/m}^2$, length of chamber $\ell = 3 \text{ m}$, cross-section areas of head end and load end $A_0 = 7 \text{ m}^2$ and $A_\ell = 0.7 \text{ m}^2$, respectively, the working gas of specific heat ratio $\gamma = 1.4$ and universal constant $R = 287.058 \text{ J/kg K}$, leakage factor $\beta = 0.9$, and working temperature $T = 300 \text{ K}$ in average. The mechatronic load is represented by the inertia of 10 kg series-connected by the damping of 4 nt s/m .

Fig. 2 shows working and source cycles coordinated by A_t/ℓ and u , wherein the first plot is without resonant control and the second plot is at resonance of the first-mode frequency $f_1 = 14.64 \text{ Hz}$. Therein both cycles are directed clockwise, meaning that the engine is doing positive work on the load end and the boundary heat-flux is exciting positive useful energy. The closed region enveloped by the work cycle has the area A_W meaning the mechanical work done by the engine over a cycle. Similarly, the enclosed area by the source cycle A_S is the useful energy triggered by the heat-flux q . The first plot verifies the conservation law presented in Eq. (42):

$$A_W = \beta A_S,$$

where $\beta = 0.9$ is the leakage factor indicating the energy loss in practical design. We capture this energy-loss effect by modelling Robin-type boundary condition upon the load end. In both cycles, the I–II phases indicate the expansion strokes, and the III–IV phases indicate the contraction strokes.

As indicated by the second plot, while the working frequency is approaching to resonant frequency, the engine cycles are getting extreme in shapes and thereby in power rating, which demonstrate the main difference between active design and passive design. For clearer view of the engine cycles at resonance, Fig. 3 shows its responses in time domain, where p_0, p_ℓ stand for the acoustic pressures at the head end and at the tail end, respectively, and v for the piston speed. With the first-mode resonance combined by the ratio $A_\ell/A_0 = 1/10$, the power is transferred from the head end with $A_0/v_0 = 1837.21 \text{ kg m/s}$ and $u_0 = 0.0014 \text{ m/s}$ of peak values to the load end with $A_\ell/v_\ell = 67.80 \text{ kg m/s}$ and $v = 6.78 \text{ m/s}$ of peak values.

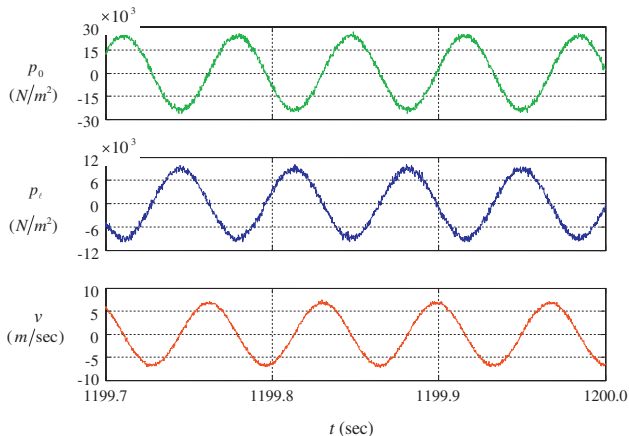


Fig. 3. Steady-state responses of head pressure, tail pressure and piston velocity at resonance.

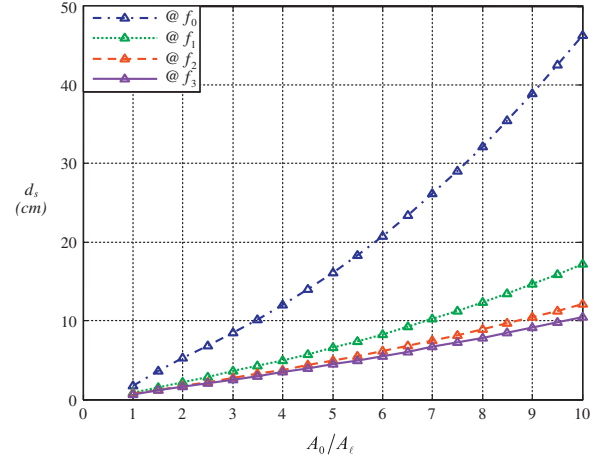


Fig. 4. Piston strokes at different resonant frequencies and head-to-tail area ratios.

Analogously, AC power is transmitted by much larger pressure and tiny velocity to the load end with proper pressure and velocity, so that the viscosity effect inside the Rijke tube is significantly reduced. That is, the ratio of the heat area A_0 to the load area A_ℓ functions like AC-power transformer in power transmission. Meanwhile, it results in long piston-stroke, which makes possible in mechanism design large gear-ratio transmission from mechanical thrust to electricity generation, and thus increases the power-generation efficiency.

Figs. 4 and 5 record the responses of piston stroke and power rating, respectively, as the photo valve is switched at the zero, first, second and third -mode frequencies and the area-ratio A_0/A_ℓ is set from 1 to 10. Fig. 4 shows that the piston stroke is boosted by increasing the area-ratio A_0/A_ℓ . For example, at first-mode resonance, the piston stroke d_s is boosted from 0.12 cm to 16.75 cm as the area-ratio A_0/A_ℓ is changed from 1 to 10. Due to the boundary friction arising from flow leakage, resonance at higher frequencies yields shorter piston-strokes with a specified area-ratio. As a result, the area-ratio in thermoacoustic engine plays analogously as gear-ratio in power-train transmission to regulate force-velocity.

Fig. 5 contains the Bode plot of p_0/p_{atm} , which shows the amplitude of head pressure p_0 and its phase difference θ between the head velocity u_0 , across the span of working frequencies. The effect of resonance on power rating can be easily examined by this Bode plot. In the magnitude plot, four resonant frequencies are marked: $f_0 = 5.37 \text{ Hz}$, $f_1 = 14.64 \text{ Hz}$, $f_2 = 24.30 \text{ Hz}$ and $f_3 = 34.29 \text{ Hz}$, at each of

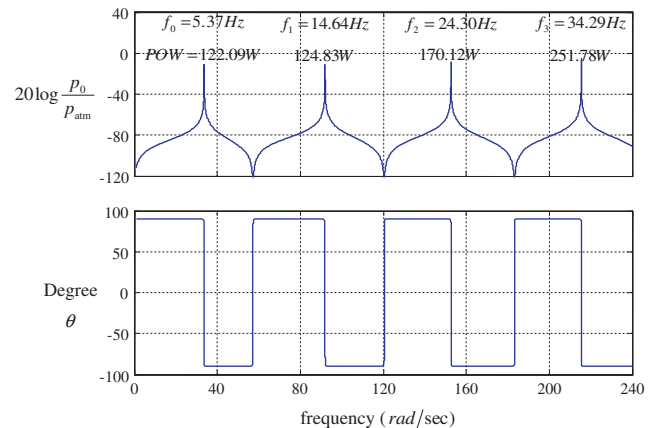


Fig. 5. Frequency response of the head pressure and power rating assessment at different resonant frequencies.

which the real power POW generated by sunlight is marked above each peaks. The real power POW is calculated by

$$\text{POW} = \beta U_0 P_0 \cos \theta,$$

where U_0 and P_0 are root-mean-square values of u_0 and p_0 , respectively, and the phase difference θ becomes the power factor in AC power. Generally, the power rating is getting larger along with resonance at higher frequencies, but the piston stroke is getting shorter.

Suppose that the head area is $A_0 = 7 \text{ m}^2$ and the efficiency of electricity generation is 60%, resonant control working at the second-mode frequency can harvest 102.07 W. A suburban family needs almost 1.0 KW to constant power it up. For any homes possessing more than 70 m^2 rooftop or land spaces, the resonant thermoacoustic engine under investigation is the best candidate served for solar energy generation to power them up or turns electricity metres backward.

6. Conclusion

This paper develops the energy-transfer modelling of active thermoacoustic engines and performs assessment of power ratings. We find that the power rating and piston stroke can be remarkably amplified through resonant control of boundary heat-flux and by shaping the resonator, respectively. Such findings are by way of Lagrange thermoacoustic dynamics, spatiotemporal transfer-function, and the conservation law of thermoacoustic storage.

Acknowledgement

National Science Council of Taiwan financially supported this work for three years. The authors here express their utmost thanks to NSC Control Group.

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